If and only if



If and only if statements

A theorem that asserts that two statements P and Q are equivalent requires you to prove *both* $P \implies Q$ and $Q \implies P$.

Chapter 7 exercise 3.

Proposition: If a is an integer, then a is even if and only if $a^3 + a^2 + a$ is even.

There are two claims:

a even implies $a^3 + a^2 + a$ is even.

$a^3 + a^2 + a$ is even implies a is even.

Each requires proof.

Proof: First we show that, if a is even, then $a^3 + a^2 + a$ is even.

Assume a is even. Then $a = 2k$ for some integer k.

Then $a^3 + a^2 + a = (2k)^3 + (2k)^2 + 2k$

$= 2(4k^3 + 2k^2 + k)$

Since $4k^3 + 2k^2 + k$ is an integer we see that

$a^3 + a^2 + a$ is even.

$P \Rightarrow Q$ is equiv. to $\neg Q \Rightarrow \neg P$

Now we show that, if $a^3 + a^2 + a$ is even, then a is even.

We will argue the contrapositive. Assume a is odd. Therefore $a = 2k+1$ for some integer k.

Then $a^3 + a^2 + a = (2k+1)^3 + (2k+1)^2 + 2k+1$

Notice that $2k+1 \equiv 1 \mod 2$

and $(2k+1)^2 \equiv 1 \mod 2$

and $(2k+1)^3 \equiv 1 \mod 2$

so $(2k+1)^3 + (2k+1)^2 + (2k+1) \equiv 1+1+1 \mod 2$

$\equiv 3 \mod 2$

$\equiv 1 \mod 2$

Thus $(2k+1)^3 + (2k+1)^2 + (2k+1) = 1 + 2j$ for some j

Suppose $x \in \mathbb{Z}$ Prove $2x+1$ is even $\Rightarrow$ x is even.

If x is odd then $2x+1$ is odd

Chapter 7, exercise 9.

Proposition: Suppose that a is an integer. Then $14|a$ if and only if $7|a$ and $2|a$.

If $14|a$, then since $7|14$ we have $7|a$ and since $2|14$ we have $2|a$ (using $a|b$ and $b|c \Rightarrow a|c$)

$12|a \Leftarrow$ ~~$2|a$ and $6|a$~~ $2|a$ and $6|a$

If $a|bc$ and $\gcd(a,b)=1$ then $a|c$.

Suppose $7|a$ then $a = 7k$

Now if $2|a$ then $2|7k$

Proof: First we suppose that $14|a$ and show that both 7 and 2 divide a.

Now we show that, if both 7 and 2 divide a, then 14 divides a.

Equivalence

Theorem Suppose A is an $n \times n$ matrix. The following statements are equivalent:

(a) The matrix A is invertible.
(b) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.
(c) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(d) The reduced row echelon form of A is I_n.
(e) $\det(A) \neq 0$.
(f) The matrix A does not have 0 as an eigenvalue.

Figure 1: Theorem from page 149

Cycle proofs

If each step in the circle of implications:

$$P_1 \implies P_2 \implies P_3 \implies \cdots \implies P_n \implies P_1$$

is true, then all of the statements are equivalent – that is, all true or all false together.